WEAK APPROXIMATION FOR LOW DEGREE DEL PEZZO SURFACES

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Abstract. Let $K = K(C)$ be the function field of a smooth curve C. Applying the result of [Xu08], we prove that if S/K is a degree one or two del Pezzo surface which can be completed to a generic family in the parametrizing space over C, then weak approximation holds at every place $c \in C$.

Contents

1. Introduction

Throughout this paper, the ground field is always of charactersitic 0. If a variety X is defined over K a number field, it is a classical question to study the existence and distribution of K-points on X. We say that X satisfies weak approximation if for any finite set of places of K and points of X over the completion of K at these places, there exists a K-rational point of X which is arbitrarily close to these points. In this note, we study varieties defined over the function field $F = K(C)$ of a smooth curve C instead of a number field. In this context, rational points correspond to sections of fibrations over a curve, and proving weak approximation corresponds to finding sections with prescribed jet data in a finite number of fibers.

The existence of sections of rationally connected fibrations was proven by Graber, Harris and Starr in [GHS03]. Kollár, Miyaoka and Mori proved the existence of

sections through a finite set of prescribed points in smooth fibers (cf. [KMM92b], 2.13 and [Ko96], IV.6.10). The existence of sections with prescribed finite jet data through smooth fibers, i.e. weak approximation at places of good reduction, was proven by Hassett and Tschinkel in [HT06]. In the same paper Hassett and Tschinkel made the following conjecture

1.1. **Conjecture.** *A smooth rationally connected variety X defined over the function field F satisfies weak approximation at places of bad reduction.*

Colliot-Thélène and Gille proved that conic bundles over $\mathbb{P}^1$ and del Pezzo surfaces of degree at least four satisfy weak approximation at all places. The cases of del Pezzo surfaces of degree less than four are still open. It is known that cubic surfaces with square-free discriminant satisfy weak approximation even at places of bad reduction (cf. [HT08]). And a similar result is generalized to degree 2 del Pezzo surfaces by Knecht [Kn08]. This paper addresses weak approximation of more case of low degree del Pezzo surfaces, including cases of degree 1 del Pezzo surfaces.

We first explain some terminology. Let S/K be a smooth del Pezzo surface of degree 1, then S can be embedded in the weighted projective space $\mathbb{P}_K(1,1,2,3)$ as a degree 6 hypersurface. So if K is the fraction field of a smooth curve C/k, we can complete S to be a family of degree 6 hypersurfaces $\mathcal{S}$ in $\mathbb{P}_C(1,1,2,3)$. We denote by $\mathbb{P}^N$ the space which parametrizes all degree 6 hypersurfaces in $\mathbb{P}(1,1,2,3)$. Because $\mathbb{P}(1,1,2,3)$ has two singular points $(0,0,1,0)$ and $(0,0,0,1)$ which are quotient singularities and of type $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, the locus parametrizing the singular surfaces consists of 3 irreducible components: two hyperplanes H_1, H_2 parametrizing hypersurfaces containing $(0,0,1,0)$ and $(0,0,0,1)$, and a hypersurface A which is the closure of the discriminant divisor A^*, where A^* parametrizes singular degree 6 hypersurfaces which does not contain $(0,0,1,0)$ nor $(0,0,0,1)$. H_1 (resp. H_2) has a dense open set H_1^0 (resp. H_2^0) parametrizing surfaces with only one singularity which is quotient and of type $\frac{1}{4}(1,1)$ (resp. $\frac{1}{9}(1,2)$) and A^* has a dense open set A^0 parametrizing the normal surfaces (see Section 3 for more details of the computation). Then

1.2. **Definition.** C is a smooth curve over k. We say that $S/K(C)$ admits a model $\mathcal{S}/C$ which is *a generic family of degree 1 del Pezzo surfaces* over C, if the morphism $K(C) \to \mathbb{P}^N$ can be completed to a morphism $f : C \to \mathbb{P}^N$ such that:

(1) C meets H_1 (resp. H_2) transversally in H_1^0 (resp. H_2^0); and
(2) C intersects A in A^0 and meets each branch of A^0 transversally.

1.3. **Theorem.** *If $S/K(C)$ is a smooth degree 1 del Pezzo surface which admits a model giving a generic family of degree 1 del Pezzo surfaces over C, then weak approximation holds at each place $c \in C$.*

1.4. **Remark.** Applying the approach in this note to the simpler case of degree 2 del Pezzo surfaces, we can prove the following result: all such surfaces are embedded in the weighted projective space $\mathbb{P}(1,1,1,2)$. So the locus parametrizing

singular fibers consists of two components: H for surfaces containing $(0, 0, 0, 1)$ and A the discrimant. We can similarly define H^0, A^* and *a generic family* of degree 2 del Pezzo surfaces over C. Then the above statement for this case is also true. We leave the details to the reader.

Now let us explain our approach. In [HT06] and [HT08], Hassett and Tschinkel initiated the method of establishing weak approximation by showing the strong rational connectedness of the smooth locus of the special fibers. In [Xu08], we show that the smooth loci of log del Pezzo surfaces are always rationally connected. However, for a given low degree del Pezzo surface S/K, usually we can not require the existence of a smooth model such that the fibers are all log del Pezzo surfaces. We have to resolve the singularities, thus there are more than one irreducible components in the special fiber. To deform a section to another one with the prescribed jet data, we have to find some auxiliary curves to correct the intersection numbers. The similar technique was applied in [HT] to study weak approximation for places where the fibers only contain ordinary singularities.

1.5. **Remark.** In [Co96], Corti established a theory of *good models* for del Pezzo surfaces. We notice that if $\mathcal{S}/C$ is a generic family in our sense, then it gives a good model over each point $c \in C$. In fact, from the local rigidity ([Co96], Theorem 1.18), we know the model we are dealing with in this note is the 'best' model. It is natural to ask whether a similar approach works for more good models

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2. Weak Approximation and Rational Curves

In this section, we will briefly recall the background. For more discussions, see [HT06] and [HT08].

2.1. **Weak Approximation.**

2.1. **Definition.** Let F be a global field, i.e, a number field or the function field of a curve C defined over an algebraically closed field k. Let S a finite set of places of F containing the archimedean places, $o_{F;S}$ the corresponding ring of integers, and $\mathbb{A}_{F;S}$ the restricted direct product over all places outside S. Let X be an algebraic variety over F, $X(F)$ the set of F-rational points and $X(\mathbb{A}_{F;S}) \subset \prod_{v \notin S} X(F_v)$ the set of $\mathbb{A}_{F;S}$-points of X. The set $X(\mathbb{A}_{F;S})$ carries a natural direct product topology. One says that *weak approximation holds for X away from S* if $X(F)$ is dense in this topology.

For our setting, to show X/K satisfies weak approximation away from S, it suffices to show for each place $c \notin S$, weak approximation holds there, i.e., we can work once at a place.

2.2. **Strong rational connectedness.** The following concept is a variant of rational connectedness in the case that the variety is smooth but nonproper.

2.2. **Definition.** ([HT08], 14) If X is smooth, then X is called *strongly rationally connected* if for each point $x \in X$ there is a morphism $f : \mathbb{P}^1 \to X^{vf}$ such that:

(1) $x \in X^{vf}$;
(2) $f^*(T_X)$ is ample.

The relationship between weak approximation and strong rational connectedness is founded by the following theorem due to Hassett and Tschinkel,

2.3. **Theorem.** *(*[HT06]*,* [HT08]*) Let X be a smooth proper rationally connected variety over $F = K(C)$, where C is a smooth curve, $B = \overline{C}/C$. Let $\pi : \mathcal{X} \to C$ a proper model of X. Let $\mathcal{X}^{sm}$ be the locus where π is smooth and $\mathcal{X}^0 \subset \mathcal{X}^{sm}$ be an open subset such that*

(1) *there exists a section $s : C \to \mathcal{X}^0$;*
(2) *for each $c \in C$ and $x \in X_c^0$, there exists a rational curve $f : \mathbb{P}^1 \to \mathcal{X}_c^0$ containing x and the generic point of $\mathcal{X}_c^0$.*

Then sections of $\mathcal{X}^0 \to C$ satisfy approximation away from B.

For strong rational connectedness of surfaces, we know the following result:

2.4. **Theorem.** [Xu08] *Let S be a log Del Pezzo surface, i.e., S only has quotient singularities and K_S is anti-ample. Then its smooth locus S^{sm} is strongly rationally connected.*

On the other hand, we also have

2.5. **Lemma.** *A point $c \in C \cap A$ satisfies the assumption (2) of (1.2) if and only if $\mathcal{S}$ is smooth and the fiber $\mathcal{S}_c$ contains at worst Du Val singularities.*

Proof. See the first paragraph of the proof of ([HT08], 23). □

Putting these two results together, for points in A, we have

2.6. **Corollary.** *Let $S/K(C)$ be a smooth log del Pezzo surface of degree 1, where C is a smooth curve. Assume S admits a model giving a generic family over C. c is a point in $C \cap A^0$. Then S satisfies weak appoximation at the place c.*

2.3. **Deformation Theory.** In our discussion, we need to apply the technique of smoothing a nodal curve to a smooth curve. This was first used in [KMM92b], and then improved in [GHS03]. For the proof of the statements in this section, see [HT06], [HT08] and [HT].

2.7. **Definition.** A projective nodal curve C is *tree-like* if

(1) each irreducible component of C is smooth; and
(2) the dual graph of C is a tree.

Let $f : C \to Y$ denote an immersion whose image is a nodal curve. The restriction homomorphism $f^*\Omega^1_Y \to \Omega^1_C$ is surjective and the dual to its kernel is still locally free. This is denoted by $\mathcal{N}_f$ and coincides with $\mathcal{N}_{C/Y}$ when f is an embedding. First order deformations of $f : C \to Y$ are given by $H^0(C, \mathcal{N}_f)$; obstructions are given by $H^1(C, \mathcal{N}_f)$. When D is a union of irreducible components of C as above, then the analogous extension takes the form:

$$0 \to \mathcal{N}_f|D \to \mathcal{N}_f \otimes \mathcal{O}_D \to Q \to 0.$$

Here Q is a torsion sheaf supported on the locus where $D^c := \overline{C \setminus D}$, with length one at each point in the locus.

2.8. **Proposition** ([HT06], Proposition 24)**.** *Let C be a tree-like curve, Y a smooth algebraic space, and $f : C \to Y$ an immersion with nodal image. Suppose that for each irreducible component C_l of C, $H^1(C_l, \mathcal{N}_f \otimes \mathcal{O}_{C_l}) = 0$ and $\mathcal{N}_f \otimes \mathcal{O}_{C_l}$ is globally generated. Then $f : C \to Y$ deforms to an immersion of a smooth curve into Y. Suppose furthermore that $P = \{p_1, ..., p_w\} \subset C$ is a collection of smooth points such that for each component C_l $H^1(\mathcal{N}_f \otimes \mathcal{O}_{C_l}(-p)) = 0$ and the sheaf $\mathcal{N}_f \otimes \mathcal{O}_{C_l}(-p)$ is globally generated. Then $f : C \to Y$ deforms to an immersion of a smooth curve into Y containing $f(p)$.*

3. The Proof of (1.3)

In this section, we study the places in H_1 and H_2. We first introduce some standard notation for quotient singularities:

3.1. **Quotient singularity.** A singularity is written as $\frac{1}{r}(a_1, a_2, ..., a_r)$ if étale locally it is given by the quotient of the action $\mathbb{Z}/r$ on $\mathbb{A}^n$ by

$$(x_1, x_2, ..., x_r) \to (\xi^{a_1}x_1, \xi^{a_2}x_2, ..., \xi^{a_r}x_r),$$

where ξ is a primitive r-root. We also assume $\gcd\{a_1, ..., a_r, r\} = 1$.
For a place $c \in C \cap H_1$, because of the assumption (1.2.1), we know that the global family $\mathcal{S}$ over $\mathcal{O}_{c,C}$ has a unique quotient singularity of type $\frac{1}{2}(1,1,1)$. Furthermore, if we write the coordinates of $\mathbb{P}(1,1,2,3)$ as (x,y,z,w), a point in H_1 gives a surface S with an equation without the term z^3. So a general equation in this form has a unique singularity at $(0,0,1,0)$. Dividing the homogeneous equation by z, the lowest non-weighted degree term has the form

$$wf_1(x,y) + f_2(x,y) = 0.$$

Here f_1 (resp. f_2) is a general linear (resp. quadratic) form of variables x and y. So the singularity is of the form

$$(z^2 = xy \subset \mathbb{A}^3)/(x,y,z) \to (-x,-y,-z),$$

which is of type $\frac{1}{4}(1,1)$. A single blow-up of $\mathcal{S}$ at p gives a resolution $\pi : \mathcal{T} \to \mathcal{S}$, with an exceptional divisor $E_1 \cong \mathbb{P}^2$, whose normal bundle is isomorphic to $\mathcal{O}(-2)$. Let E_0 be the birational transform of S_c, then $\pi|E_0 : E_0 \to S_c$ is the minimal resolution of S_c, with the exceptional curve $R = E_0 \cap E_1$ with self-intersection -4.

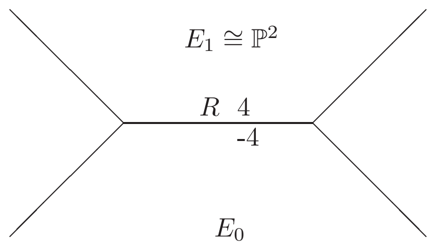

figure 1

A similar computation shows that a general point in H_2 parametrizes a surface S with unique singularity of type $\frac{1}{9}(1,2)$. After doing the 'ecconomic resolution' of the local model as in [Re87], the fiber consists of 3 components: the birational transform E_0 of S_c and two exceptional divisors $E_1 \cong \mathbb{P}^2$, $E_2 \cong F_2$. Furthermore, we have $E_0 \cap E_1 = R_1$ is the (-2)-curve in E_0 and a line in E_1; $E_0 \cap E_2 = R_2$ is the (-5)-curve on E_0 and of the class $e+3f$ (e is class of the section with negative self-intersection and f is the class of the fibers) in R_2 and $E_1 \cap E_2 = R_3$ is a line in E_1 and the section of negative self-intersection in E_2.

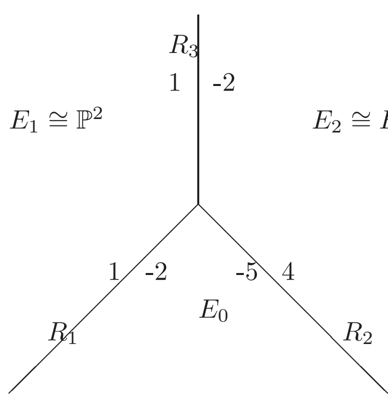

figure 2

3.2. **Auxiliary Curve.** To deform a section from one component to another given component, we need to attach auxiliary curves in E_0 on the original section to correct the intersection numbers. More precisely we show that

3.1. **Proposition.** *In the above two cases, for any exceptional curve $R \subset E_0 \to S_c$, there exists a rational curve $f : \mathbb{P}^1 \to E_0$ meeting R transversally in one point and avoiding the other exceptional curves.*

3.2. **Remark.** Besides the strong rational connectedness of S^{sm}, this is another part of Hypothesis 14 (Key Hypothesis) in [HT] when they deal with ordinary singularities.

Proof. The idea of the following elementary (but slightly tedious) argument is to show that given all these numerical data, the possible configuration of the exceptional curves is simple, partly due to we have

($*$) the only curves intersecting K_{E_0} nonnegatively are the exceptional curves.

(I): S_c contains a unique singularity of the type $\frac{1}{4}(1,1)$. In this case, E_0 is a smooth surface with $K_{E_0}^2 = 0$. If E_0 admits a morphism to a Hirzebruch surface $F_n (n \geq 2)$, then the image of R in F_n is the unique section with negative self-intersection. So we can choose f to be the birational transform of a general fiber of F_n. Thus in the following we will assume that there exists a morphisim $g : E_0 \to \mathbb{P}^2$. We can also assume E is not contracted by g. Let D_i be the class of the exceptional curves of g. (Because of ($*$), g is the blowing up of 9 distinct points on $\mathbb{P}^2$). Then the class of R is

$$g^*\mathcal{O}(n) - a_1 D_1 - \cdots - a_9 D_9 \text{ with } a_1 \geq \cdots \geq a_9 \geq 0.$$

and $K_{E_0} \backsim g^*\mathcal{O}(3) + D_1 + \cdots + D_9$. Computing R^2 and $R \cdot K_{E_0}$, we have

$$n^2 - a_1^2 - \cdots - a_9^2 = -4 \text{ and } -3n + a_1 + \cdots + a_9 = 2.$$

Then we have the equation

$$9(a_1^2 + \cdots + a_9^2) = (a_1 + \cdots + a_9)^2 - 4(a_1 + \cdots + a_9) + 40.$$

The only solutions in nonnegative integers are

$$(a_1, \ldots, a_9) = (1, \ldots, 1, 0) \text{ or } (1, 1, 1, 1, 0, \ldots, 0),$$

and it is easy to get the conclusion in these cases.
(II): S_c contains a unique singularity of type $\frac{1}{9}(1,2)$. In this case, the basic idea is similar, but it requires more analysis. First, we observe that

3.3. **Lemma.** *It suffices to find the rational curve for one of $R_i (i = 1, 2)$.*

Proof. Applying the strong rational connectedness of $E^0 \setminus \{R_1 \cup R_2\}$ to adding enough teeth, we can assume our rational curve to be very free. If there is such a curve K meeting R_2, but not R_1., then we can do a small deformation and choose five such curves K_i $(1 \leq i \leq 5)$ meeting R_2 transversally on different points. Now we consider the immersion $f : C \to E_0$, where C is the comb with handle E_2 and

teeth K_i. It follows from the discussion in Section (2.3) that $\mathcal{N}_f|E_2 = \mathcal{O}$, thus we can deform the comb to get a smooth curve K' with $K' \cdot R_2 = 0$ and $F' \cdot R_1 = 1$. The argument of 'jumping' from R_1 to R_2 is the same and we will omit it. $\square$

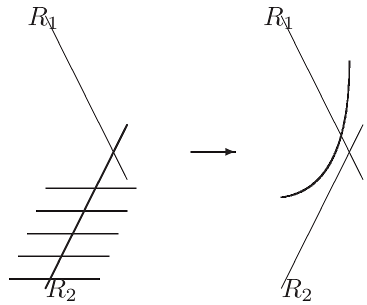

figure 3

$\square$

Let us first assume that there exists a morphism $g : E_0 \to \mathbb{P}^2$ such that none of R_1 and R_2 is contracted by g. Then let the class of R_1 be $f^*\mathcal{O}(m) - b_1D_1 - \dots b_9D_9$ and R_2 be $f^*\mathcal{O}(n) - a_1D_1 - \cdots - a_9D_9$ ($a_1 \geq \cdots \geq a_9$), we will have

$$9(a_1^2 + \cdots + a_9^2) = (a_1 + \cdots + a_9)^2 - 6(a_1 + \cdots + a_9) + 54$$

and

$$9(b_1^2 + \dots b_9^2) = (b_1 + \cdots + b_9)^2 + 18.$$

We also have $R_1 \cdot R_2 = 1$. Then an elementary computation shows that the only solution is

$$(a_1, \dots, a_9) = (1, \dots, 1, 0, 0, 0) \text{ and } (b_1, \dots, b_9) = (0, \dots, 0, 1, 1, 1).$$

In this case, it is easy to see the statement holds. If g contracts at least one of R_1 or R_2, then we can assume that g does not contract both of them. So if g contacts R_1, we have

$$(a_1, \dots, a_9) = (1, \dots, 1) \text{ or } (1, \dots, 1, 0, 0, 0).$$

Then it is easy to find a requiring free curve for R_2. If g only contracts R_2, then g will blow up a point with anther 4 points in its first infinitesimal neighborhood. By contracting other collection of -1-curves, we can easily find another morphism $g' : E_0 \to \mathbb{P}^2$ such that g' does not contract R_2, thus we reduce to the above cases. Now if $g : E_0 \to F_n$ for $n \geq 2$, then the birational transform of the section B with negative self-intersection is R_i (i=1 or 2). We can assume g does not contract any R_i, otherwise the birational transform of a general fiber of F_n gives us the curve. We claim that in this case, the image of R_{3-i} is contained in a fiber of F_n. Granted this, we can choose our rational curve to be the birational transform of a general fiber of F_n. To verify the claim, if the birational transform of B is R_2, then by changing the model, we can indeed assume $n = 5$. Then the class of R_1 is

$$g^*(ae + (5a + 1)f) + a_1E_1 + \dots a_8E_8,$$

where e is the class of B, f is the class of the fiber of F_n, and E_i is the exceptional curves of g. Knowing $R_1^2 = -2$, $R_1 \cdot K_{E_0} = 0$ and

$$8(a_1^2 + \cdots + a_8^2) \geq (a_1 + \ldots a_8)^2,$$

we have the inequality $8(5a^2 + 2a + 2) \geq (7a + 2)^2$, which implies $a = 0$. The argument for the case that $B = g(R_1)$ is similar, and we leave it to the reader. □

3.4. **Remark.** In general, there is obstruction in the Neron-Severi group for the existence of such auxiliary curves. In particular, in [HT] the authors observed that for the Cayley cubic surface

$$S : xyz + yzw + zwx + wxy = 0,$$

if we take the minimal resolution T, then for any E_i an exceptional divisor, there does not exist any curve which meets E_i transversally at one point but avoids other exceptional curves, because the sum of all E_i is divided by 2 in the Neron-Severi group.

3.3. **Moving Sections.** In this subsection, we will explain the procedure how we start from a section, by attaching rational curves on the special fiber T_c, we can deform it to a new section with a given prescribed jet data. In fact, our argument is similar to the one in [HT06] and [HT], but with different configuration of divisors in the special fiber. We will do the harder case for places in H_2 and leave the argument for places in H_1 to the reader.

Step 1: Moving sections to E_0.
In this step, we want to show that there always exists a section meeting the T_c in E_0.
Given a section C, by attaching enough free curves in general fibers we can assume it is free. If it meets the special fiber in E_1. Then choose C_1 a line passing the intersection of C with E_1 which meets R_1 and R_3 at general points; C_2 the ruling of R_2; and C_3 the curve given by (3.1) meeting R_2 transversally at the point where C_2 meets. Gluing C and C_i $(1 \leq i \leq 3)$ together in the obvious way, we get a tree-like curve $f : \overline{C} \to \mathcal{T}$. Since $\mathcal{N}_f|_{C_1} \cong \mathcal{O} \oplus \mathcal{O}(1)$ and $\mathcal{N}_f|_{C_2} \cong \mathcal{O} \oplus \mathcal{O}$, it following from (2.8), that we can deform $\overline{C}$ to a smooth curve, which is a section meeting E_0 by computing the intersection number.

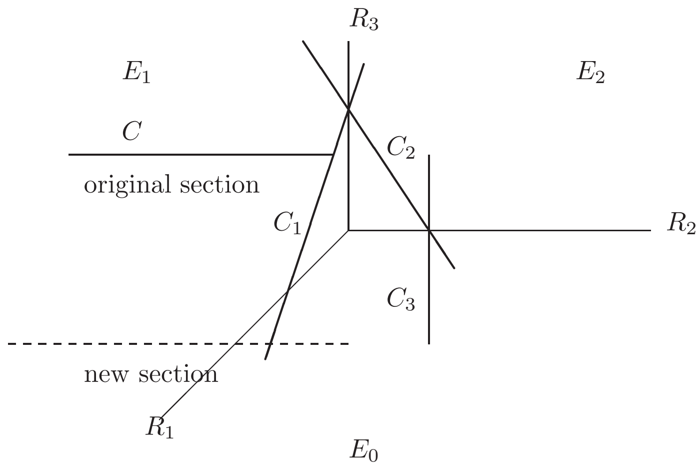

figure 4

If the section C meets E_2, then we choose a general curve C_1 with the class $e+2f$ in E_2, which meets R_2 at three general points but not R_3. Applying (3.1), we can choose free curves C_2 and C_3 in T_c which meets two of these three points. Similarly, we can glue C and C_i $(1 \leq i \leq 3)$ together to get a tree-like curve, which can be deformed to a smooth section meeting E_0.

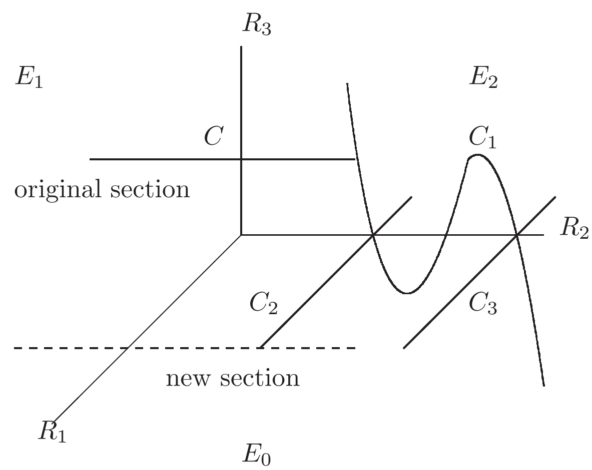

figure 5

Step 2: Moving sections out of E_0.
In this step, we show that for any component E_i of T_c, there is a section which meets E_i.
Let C_1 (resp. C_2) be a curve in E_0 meeting R_1 (resp. R_2) transversally at one point and contains the point where C meets E_0. The existence of C_1 and C_2 is proved in (3.1). Gluing the curves together, a similar computation shows that we can smooth it. Then the computation of the intersection number shows that the smoothing curve is a section meeting E_1.

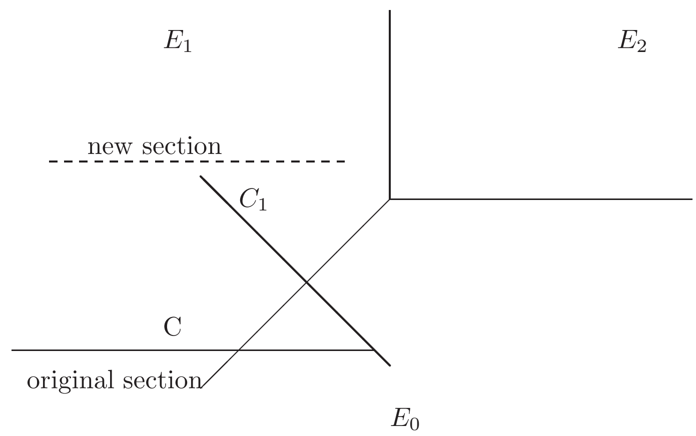

figure 6

Step 3: Moving sections in the same component
We apply ([HT], Proposition 22) in this step. Thus it suffices to check that for E_i and a point $p \in E_i^{sm}$, we can find a connected nodal curve K of genus zero, distinguished smooth points $0, \infty \in K$ in the same irreducible component, and a differential-geometric immersion h mapping K in to the special fiber with the following properties

(1) $h(0) = p$ and $h(\infty) = q$ a general point in E_i^{sm};
(2) each irreducible component of the special fiber intersects C with degree zero;
(3) h takes K to the open subset of the special fiber with normal crossings singularities of multiplicity at most two; $f^{-1}(\cup R_i) \subset K^{sing}$ and at points of $R_i \cap h(K)$ $(1 \leq i \leq 3)$ there is one branch of K through each component of the special fiber; and
(4) $\mathcal{N}_h(-p)$ is globally generated.

For E_1, the reducible curve $K = \cup_{1 \leq i \leq 4} C_i$ depicted in figure 7 will give the curve as above, where C_1 is a line containing p and q; C_2 is a ruling of R_2; C_3 (resp. C_4) is given by (3.1) which meets R_1 (resp. R_2) transversally at one point but not R_2 (resp. R_1).

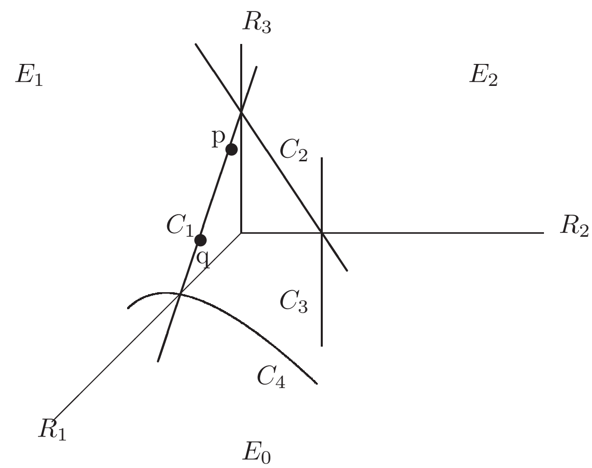

figure 7

Similarly for E_2, the reducible curve $K = \cup_{1 \leq i \leq 4} C_i$ depicted in figure 8 gives the curve which we are looking for. C_1 is a curve with class $e + 2f$ containing p and q; C_2, C_3 and C_4 are three curves given by (3.1) meeting R_2 transversally at one point but not R_1.

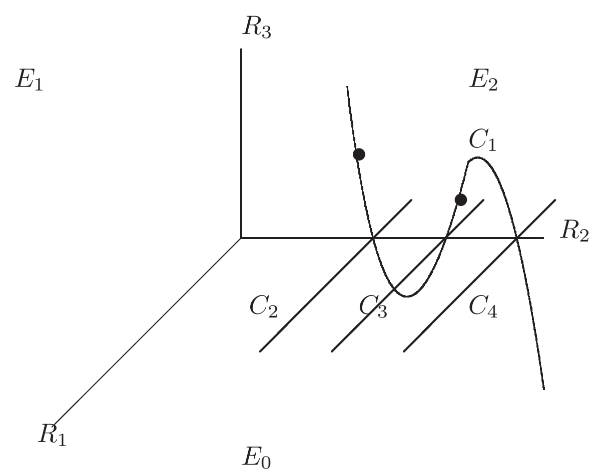

figure 8

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